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Orbital Aharonov–Anandan geometric phase for confined motion in a precessing magnetic field

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Abstract. We obtain general formulae to calculate exactly the orbital Aharonov–Anandan phase for a bounded system interacting with a precessing magnetic field in the case where the binding potential is rotationally invariant around the axis of rotation of the magnetic field. The particular case of a quadratic potential is considered in full detail, and the connection with the stability problem is established. The adiabatic limit of the Aharonov–Anandan phase to obtain the corresponding Berry phase is also considered.

1. Introduction

As is well known, the Hilbert space of states of any quantum system $\mathcal{H}$ can be seen as a fibre bundle with base space the projective state space and fibre U(1). There is a canonical connection in this fibre bundle [1–4], which allows one to lift curves in the base into horizontal curves in the fibre bundle; the horizontal lift in $\mathcal{H}$ of a closed path in the base is in general open. The curve starts and ends in the same fibre but at different points, and the holonomy of the path is the Aharonov–Anandan (AA) geometric phase [1]. This geometric phase is associated to any closed path in the projective state space, regardless of whether or not it is an actual evolution curve for the system. In general relativity, there is also an holonomy associated to closed circuits in spacetime, but these circuits cannot be made of a single closed future time-like geodesic.

The situation for the holonomy described by geometric phases (both general and adiabatic [5–7]) is different, because it could happen that the actual evolution of a system follows a closed curve in the projective state space. This is the case for cyclic states, where a quantum system evolves under the action of a Hamiltonian in such a way that after a lapse of time T it returns to its initial state, i.e. $|\psi(T)\rangle = e^{i\phi}|\psi(0)\rangle$. These states allow a direct experimental study of geometric phases, because a closed path of the system is simply provided by time evolution.

In this paper we study orbital geometric phases for a system with cyclic states: a charged spinless particle in a Hamiltonian of the type

$$H(t) = \frac{1}{2m}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}(\boldsymbol{r},t)\right)^2 + V(\boldsymbol{r}) \tag{1.1}$$

where $\boldsymbol{A}(\boldsymbol{r},t)$ is the vector potential associated to a rotating magnetic field and $V(\boldsymbol{r})$ is a binding potential with rotational invariance around the axis of rotation of the

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magnetic field (say, the x_3-axis). We will obtain explicit expressions for the AA phase, by transforming the original time-dependent problem into a time-independent one in the rotating frame. If the new time-independent 'Hamiltonian' possesses bounded states, then we will show that the corresponding states are cyclic, and that their AA phases are proportional to the expectation values of the third component of the angular momentum in these states. Moreover, these phases will be related to the corresponding 'energy' eigenvalues in the rotating frame. This procedure can be fully carried out in the case of a binding quadratic potential (including zero), and the system presents two different behaviours according to the values of some parameters. In one case the motion is confined and there exist cyclic evolutions while in the other the trajectories are deconfined and it is not possible to obtain a cyclic evolution. Regarding the confinement regions for these kinds of systems, the reader may refer to the works of Mielnik and Fernández-C [8–11] and references therein.

This complements a recent paper [12], where we have studied geometrical AA phases for $\frac{1}{2}$-spin systems evolving under periodic time-dependent magnetic fields, and their adiabatic limits; Berry's phase is recovered as a first-order term, the remaining ones describing deviations from adiabaticity. We also consider here the behaviour of AA phases for the system (1.1) in the adiabatic limit. In order to recover Berry's phases, by making the precession frequency go to zero, some care must be exercised. This is so because in this limit the system could be brought outside the confinement domain; this is particularly evident when $V = 0$. In this case, even though in the non-adiabatic situation the charged particle could be trapped by the precessing magnetic field, when $\omega \to 0$ the motion along the magnetic field direction becomes free, and we do not have cyclic states at all (although one degree of freedom still produces a cyclic motion). If taken carefully, however, the adiabatic limit of the AA phase turns out to be the 'correct way' to calculate the corresponding Berry phase.

The paper is organized as follows. In section 2 we present a general analysis to evaluate AA phases for a system described by a Hamiltonian of the form (1.1). The case in which V is a binding quadratic potential is analysed in section 3, where we also establish the conection between the existence of the bounded (cyclic) states and the existence of the stability (trapping) regions for the corresponding dynamical systems. In section 4, we discuss on the stability and instability regions for some physically interesting quadratic examples and the adiabatic limit of their corresponding AA phases.

2. AA phase in a precessing magnetic field

Let a charged spinless particle interact with a precessing magnetic field described by the generic Hamiltonian (1.1), where $e < 0$, $\boldsymbol{A}(\boldsymbol{r}, t) = -\frac{1}{2}\boldsymbol{r} \times \boldsymbol{B}(t)$ and

$$\boldsymbol{B}(t) = (B\cos\omega t, B\sin\omega t, B_0)\,. \tag{2.1}$$

Suppose that $V(\boldsymbol{r})$ is rotationally invariant around x_3-axis. Then (1.1) can be written (with $\hbar = 1$) as

$$\begin{aligned} H(t) &= \frac{\boldsymbol{p}^2}{2m} - \frac{e}{2mc}\boldsymbol{L}\cdot\boldsymbol{B}(t) + \frac{e^2}{8mc^2}\{\boldsymbol{r}^2\boldsymbol{B}(t)^2 - [\boldsymbol{r}\cdot\boldsymbol{B}(t)]^2\} + V\left(\sqrt{x_1^2 + x_2^2}, x_3\right) \\ &= \mathrm{e}^{-\mathrm{i}\omega t L_3} H(0) \mathrm{e}^{\mathrm{i}\omega t L_3}\,. \end{aligned} \tag{2.2}$$

The equation for the evolution operator $U(t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}U(t) = -\mathrm{i}H(t)U(t) \qquad U(0) = I \tag{2.3}$$

can be transformed in a time-independent equation by means of a 'transition to the rotating frame', i.e. $U(t) = \mathrm{e}^{-\mathrm{i}\omega t L_3}W(t)$ [8–10, 13–14]. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}W(t) = -\mathrm{i}GW(t) \qquad W(0) = I \tag{2.4}$$

where G appears as a time-independent Hamiltonian in the rotating frame,

$$G = H(0) - \omega L_3 . \tag{2.5}$$

Note that the operator G depends linearly and continuously on ω. The solution of (2.4) is immediate, $W(t) = \mathrm{e}^{-\mathrm{i}Gt}$, leading to

$$U(t) = \mathrm{e}^{-\mathrm{i}\omega t L_3}\mathrm{e}^{-\mathrm{i}Gt} . \tag{2.6}$$

Suppose now that the spectrum of G is discrete and non-degenerate (for values of the parameter ω around the frequency of precession of the magnetic field (2.1)), and let $\{|\psi_n\rangle\}$ be a set of normalized bounded states with eigenvalue $\mathcal{E}_n$. So we have

$$G|\psi_n\rangle = \mathcal{E}_n|\psi_n\rangle \tag{2.7}$$

where we assume that $|\psi_n\rangle$ and $\mathcal{E}_n$ depend continuously on ω. From (2.6) and (2.7), the time evolution of $|\psi_n\rangle$ is simply

$$U(t)|\psi_n\rangle = \mathrm{e}^{-\mathrm{i}\mathcal{E}_n t}\mathrm{e}^{-\mathrm{i}\omega t L_3}|\psi_n\rangle . \tag{2.8}$$

For $t = 2\pi/\omega \equiv T$, this relation reduces to

$$U(T)|\psi_n\rangle = \mathrm{e}^{-\mathrm{i}\mathcal{E}_n T}|\psi_n\rangle . \tag{2.9}$$

Hence, the bounded state $|\psi_n\rangle$ becomes cyclic and could possess a geometric phase. Following [1], the AA phase associated to any cyclic state $|\psi(T)\rangle = \mathrm{e}^{\mathrm{i}\phi}|\psi(0)\rangle$ is

$$\beta = \phi + \mathrm{i}\int_0^T \langle\psi(t)\left|\frac{\mathrm{d}}{\mathrm{d}t}\right|\psi(t)\rangle \mathrm{d}t . \tag{2.10}$$

In our case $\phi = -\mathcal{E}_n T$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}(U(t)|\psi_n\rangle) = -\mathrm{i}(\mathcal{E}_n + \omega L_3)\mathrm{e}^{-\mathrm{i}\mathcal{E}_n t}\mathrm{e}^{-\mathrm{i}\omega t L_3}|\psi_n\rangle . \tag{2.11}$$

Using (2.11) and the fact that $|\psi_n\rangle$ is time-independent, (2.10) becomes

$$\beta_n = 2\pi\langle\psi_n|L_3|\psi_n\rangle . \tag{2.12}$$

This gives a simple expression for β; however, it requires the knowledge of $\{|\psi_n\rangle\}$. Sometimes it is easier to find the eigenvalues of G, so we will transform (2.12) to an equivalent relation involving the ω dependence of $\mathcal{E}_n$ around the fixed value of ω

given in (2.1). Deriving (2.7) with respect to ω, using (2.5) and noticing that $H(0)$ does not depend on ω we get

$$-L_3|\psi_n\rangle + G\frac{\partial}{\partial\omega}|\psi_n\rangle = \frac{\partial\mathcal{E}_n}{\partial\omega}|\psi_n\rangle + \mathcal{E}_n\frac{\partial}{\partial\omega}|\psi_n\rangle\,. \tag{2.13}$$

Multiplication of (2.13) by $\langle\psi_n|$ and use of (2.7) leads to

$$\langle\psi_n|L_3|\psi_n\rangle = -\frac{\partial\mathcal{E}_n}{\partial\omega}\,. \tag{2.14}$$

Finally, inserting (2.14) into (2.12) gives the desired relation

$$\beta_n = -2\pi\frac{\partial\mathcal{E}_n}{\partial\omega}. \tag{2.15}$$

Expressions (2.14) and (2.15) are valid when the eigenvalue $\mathcal{E}_n$ is non-degenerate. Geometric phases for degenerate cases have been largely studied in the literature (see e.g. [15–16]). In our problem the same technique as that used in the above non-degenerate case can be applied even if there is degeneration; however some care must be exercised if the degeneration changes with ω. We illustrate in figure 1 some possible different cases for the ω dependence of the eigenvalues of $G(\omega)$, including an example of level crossing; this raises the question of whether level crossing is actually possible for the Hamiltonian (2.5). It is clear that levels can cross if $[H(0), L_3] = 0$, and as far as we know no general principle precludes level crossing even if $[H(0), L_3] \neq 0$. Indeed, when the theory developed in section 3 is applied to the examples in section 4, examples with crossings appear; we furnish details for a such situation in section 4 (Case A).

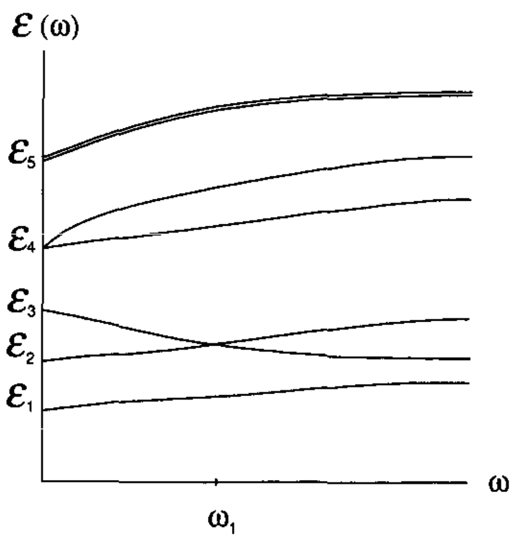

Figure 1. Possible dependence of the eigenvalues of G with ω. The two curves associated to $\mathcal{E}_5$ are superposed, but we shift one of them to illustrate double degeneracy for all ω.

Consider first the case of 'levels' $\mathcal{E}_2$ and $\mathcal{E}_3$ at $\omega = \omega_1$. For close values of ω, $\mathcal{E}_2(\omega)$ and $\mathcal{E}_3(\omega)$ are non-degenerate eigenvalues, with state vectors $|\psi_2(\omega)\rangle$ and $|\psi_3(\omega)\rangle$, which are orthogonal (even for $\omega = \omega_1$ by continuity in ω); for these we have

$$\langle\psi_i(\omega_1)|L_3|\psi_i(\omega_1)\rangle = -\left.\frac{\partial\mathcal{E}_i}{\partial\omega}\right|_{\omega=\omega_1} \qquad i = 2,3\,.$$

Furthermore, if equation (2.13) for $|\psi_2(\omega)\rangle$ is multiplied by $\langle\psi_3(\omega)|$ and the limit $\omega \to \omega_1$ is taken, one easily concludes that $\langle\psi_3(\omega_1)|L_3|\psi_2(\omega_1)\rangle = 0$. So if we refer a generic state of 'energy' $\mathcal{E}_2(\omega_1) = \mathcal{E}_3(\omega_1)$ to this basis, $|\psi\rangle = \alpha_2|\psi_2\rangle + \alpha_3|\psi_3\rangle$, the geometric phase β associated with the cyclic evolution of the initial state $|\psi\rangle$ after a time interval T will be equal to $2\pi\langle\psi|L_3|\psi\rangle$, and

$$\langle\psi|L_3|\psi\rangle = -|\alpha_2|^2\frac{\partial\mathcal{E}_2}{\partial\omega} - |\alpha_3|^2\frac{\partial\mathcal{E}_3}{\partial\omega}\,.$$

Different cyclic states in the same eigenspace will give different values of the geometric phase.

For the case where the degeneration does not change with ω (as in $\mathcal{E}_5$ in figure 1) we have a single function $\mathcal{E}_5(\omega)$; the reasoning leading to (2.14) and (2.15) is applicable without any change, with any initial vector $|\psi\rangle$ in the eigenspace of $G(\omega)$. At first sight (2.14) is somewhat surprising, because its right-hand side is independent of the specific vector $|\psi\rangle$, while there does not seems not to be any *a priori* reason which would make the left-hand side also independent of $|\psi\rangle$. However, it can be readily seen from perturbation theory (see for instance [17]) that if the degeneration does not change with ω, then $\langle\psi|L_3|\psi\rangle$ should indeed be independent of the vector $|\psi\rangle$ in the corresponding eigenspace of $G(\omega)$.

Equations (2.12) and (2.15) could serve to evaluate the geometric phases in very diverse situations. In particular, they would be useful if V is a radial binding potential (the Morse potential, the hydrogen atom potential, etc), although concrete calculations would be hard. In the following we will restrict ourselves to the simple case in which V is a quadratic form in (x_1, x_2, x_3), and where the $\mathcal{E}_n$ can be obtained directly.

3. Example: the quadratic case

The most general quadratic potential with axial symmetry around the x_3-axis reads

$$V\left(\sqrt{x_1^2 + x_2^2}, x_3\right) = V_0(x_1^2 + x_2^2) + V_1 x_3^2 \tag{3.1}$$

with V_0 and V_1 constants. The discrete spectrum of the Hamiltonian (2.5) can be easily found because G is quadratic in $\boldsymbol{u}^{\mathrm{T}} = (\boldsymbol{r}, \boldsymbol{p})$ [8–11, 18–21]:

$$\begin{aligned} G = \frac{\boldsymbol{p}^2}{2m} + bL_1 + (b_0 - \omega)L_3 + \frac{m}{2}\big[\boldsymbol{r}^2(b^2 + b_0^2) - (bx_1 + b_0x_3)^2 \\ + v_0(x_1^2 + x_2^2) + v_1x_3^2\big] \end{aligned} \tag{3.2}$$

where $b = |e|B/2mc > 0$, $b_0 = |e|B_0/2mc > 0$, $v_0 = 2V_0/m$ and $v_1 = 2V_1/m$. Following [8–10], we first study the evolution of the Heisenberg canonical trajectories in the rotating frame given by

$$u(t) = \mathrm{e}^{\mathrm{i}Gt} u \mathrm{e}^{-\mathrm{i}Gt} = \mathrm{e}^{\Lambda t} u \tag{3.3}$$

where the 6×6 matrix Λ, obtained through $[\mathrm{i}G, u] = \Lambda u$, reads

$$\Lambda = \begin{pmatrix} 0 & \omega - b_0 & 0 & 1/m & 0 & 0 \\ b_0 - \omega & 0 & -b & 0 & 1/m & 0 \\ 0 & b & 0 & 0 & 0 & 1/m \\ -m(b_0^2 + v_0) & 0 & mbb_0 & 0 & \omega - b_0 & 0 \\ 0 & -m(b^2 + b_0^2 + v_0) & 0 & b_0 - \omega & 0 & -b \\ mbb_0 & 0 & -m(b^2 + v_1) & 0 & b & 0 \end{pmatrix}. \tag{3.4}$$

Quantum as well as classical motions are determined by the roots of the characteristic polynomial of Λ

$$P(\lambda) = \mathrm{Det}(\lambda I - \Lambda) = \lambda^6 + C_1\lambda^4 + C_2\lambda^2 + C_3 \tag{3.5a}$$

where

$$\begin{aligned} C_1 &= 4b^2 + 4b_0^2 + 2v_0 + v_1 - 4b_0\omega + 2\omega^2 \\ C_2 &= 2\omega(\omega - 2b_0)(v_1 - v_0) + 4b_0\omega^2(b_0 - \omega) + \omega^2(\omega^2 + 3b^2) \\ &\qquad + v_0(v_0 + 4b^2) + 2v_1(v_0 + 2b_0^2) \end{aligned} \tag{3.5b}$$

$$C_3 = \omega(2b_0 - \omega)(2v_0v_1 + 2b_0v_1\omega - v_1\omega^2 - b^2\omega^2) + v_0(v_0v_1 - b^2\omega^2)\,.$$

For the case where Λ is diagonalizable, there exist two different behaviours for $u(t)$, which depend on the values taken by the set of four dimensionless parameters

$$\{\alpha = b/\omega,\ \ \alpha_0 = b_0/\omega,\ \ \gamma_0 = v_0/\omega^2,\ \ \gamma_1 = v_1/\omega^2\}\,. \tag{3.6}$$

(1) All the six roots of $P(\lambda)$ are non-null and imaginary, hence the spectral decomposition of $\mathrm{e}^{\Lambda t}$ will consist of the superposition of three independent oscillating motions which, according to (3.3), will induce confined motions on the canonical vector $u(t)$;
(2) At least two roots of $P(\lambda)$ have a non-null real part; then the spectral decomposition of $\mathrm{e}^{\Lambda t}$ will contain an exponential term and, therefore, the trajectories (3.3) will be, in general, deconfined.

The above cases also apply in relation with the spectrum of G. So, in the case where the parameters (3.6) fall into the confinement domain, G can be expressed as the superposition of three harmonic oscillator Hamiltonians and, therefore, the spectrum of G will be discrete. On the other hand, if the system parameters fall into a deconfinement region, at least a contribution to (3.2) will consist of a repulsive oscillator Hamiltonian and the spectrum of G will be continuous.

To obtain the AA phase (given by (2.15)) explicitly, suppose that the parameters (3.6) belong to the confinement domain; then, the eigenvalues of Λ take the form $\{\pm\mathrm{i}\omega_1, \pm\mathrm{i}\omega_2, \pm\mathrm{i}\omega_3\}$, and G can be expressed as [8, 10]

$$G = \sum_{i=1}^{3} \epsilon_i\omega_i \left(A_i^\dagger A_i + \frac{\epsilon_i}{2}\right) \tag{3.7}$$

where $A_i^\dagger = e_i u$, $A_i = e_i^* u$, and e_i are the row eigenvectors of Λ, $e_i \Lambda = \mathrm{i}\omega_i e_i$, which form a dual system to the column eigenvectors r_j of Λ, $\Lambda r_j = i\omega_j r_j$, i.e. $e_i r_j = \delta_{ij}$. The coefficients ϵ_i in (3.7) may take on the values $+1$ or -1 and come from the evaluation of the commutation relationships between the set of operators $\{A_i, A_j^\dagger, i,j = 1,2,3\}$

$$[A_i, A_j] = [A_i^\dagger, A_j^\dagger] = 0 \qquad [A_i, A_j^\dagger] = \epsilon_j \delta_{ij}. \tag{3.8}$$

Note that we are using the name 'harmonic oscillator Hamiltonian' in a broad sense. Indeed, if some of the ϵ_j takes the value -1, its corresponding ω_j-contribution to G will consist of the so-called *anti-oscillator* Hamiltonian (see for instance [20]), but we will not use that terminology here.

The discrete spectrum of G (see (3.7)) has the form

$$\mathcal{E}_{n_1 n_2 n_3} = \sum_{i=1}^{3} \epsilon_i \omega_i \left(n_i + \frac{1}{2}\right). \tag{3.9}$$

From (2.15), the geometric phase associated to the cyclic state $|n_1, n_2, n_3\rangle$ is

$$\beta = -2\pi \sum_{i=1}^{3} \epsilon_i \left(n_i + \frac{1}{2}\right) \frac{\partial \omega_i}{\partial \omega}. \tag{3.10}$$

Inserting $\lambda = \mathrm{i}\omega_i$ in (3.5) and using $P(\mathrm{i}\omega_i) = 0$, we evaluate $\partial\omega_i/\partial\omega$ in terms of ω_i. Then

$$\beta = -2\pi \sum_{i=1}^{3} \epsilon_i \left(n_i + \frac{1}{2}\right) \left(\frac{\partial C_1/\partial\omega\omega_i^4 - \partial C_2/\partial\omega\omega_i^2 + \partial C_3/\partial\omega}{2\omega_i(3\omega_i^4 - 2C_1\omega_i^2 + C_2)}\right). \tag{3.11}$$

Summarizing the results of this section, the evaluation of the geometric (AA) phases associated to the cyclic states of the Hamiltonian (2.2)–(3.1) has been reduced to the determination of the domain of the parameters (3.6) for which all the roots of $P(\lambda)$ are imaginary (confinement domain). Once this domain has been delimited, we use $P(\mathrm{i}\omega_j) = 0$ to find the ω_j and to obtain the discrete spectrum of the Hamiltonian in the rotating frame through (3.9). Finally, we insert the ω_j into (3.11) to determine β.

For completeness, we give the explicit expressions of the ω_j in terms of C_1, C_2, C_3:

$$\begin{aligned}
\omega_1 &= \sqrt{\frac{C_1}{3} + 2\left(\frac{q}{3}\right)^{1/2} \cos\frac{\phi}{3}} \\
\omega_2 &= \sqrt{\frac{C_1}{3} - 2\left(\frac{q}{3}\right)^{1/2} \cos\frac{\pi - \phi}{3}} \\
\omega_3 &= \sqrt{\frac{C_1}{3} - 2\left(\frac{q}{3}\right)^{1/2} \cos\frac{\pi + \phi}{3}}
\end{aligned} \tag{3.12}$$

where

$$\phi = \cos^{-1}\left[\left(\frac{3}{q}\right)^{3/2} \frac{r}{2}\right] > 0 \qquad q = \frac{C_1^2}{3} - C_2 \qquad r = \frac{2C_1^3}{27} - \frac{C_1 C_2}{3} + C_3. \tag{3.13}$$

4. Discussion

The confinement and deconfinement domains for some quadratic Hamiltonians of type (2.2), (3.1), (3.2) have been studied for a sequence of interesting physical arrangements [8–10]. We are going to describe briefly some cases for particular values of the potential term.

Case A. The simplest example is obtained by taking $B \neq 0$ and $B_0 = V_0 = V_1 = 0$; the system analysed is then a spinless charged particle evolving in a rotating magnetic field [8–9, 22]. The confinement domain is an interval in the α-line ($\alpha = |e|B/2mc\omega$) [8–9]:

$$0 < \alpha < \alpha_{cr} \approx 0.579\,982\,.$$

So confinement implies that for fixed B, the precession frequency should be larger than approximately $(|e|B/2mc) \times \frac{1}{0.579\,982}$. The characteristic frequencies in (3.7) depend continuously on ω and when they are labelled so that $0 < \omega_1 < \omega_2 < \omega_3$, the coefficients ϵ_i turn out to be $\epsilon_1 = 1,\ \epsilon_2 = -1,\ \epsilon_3 = 1$. The explicit expression for the level $\mathcal{E}_{n_1n_2n_3}$ is

$$\mathcal{E}_{n_1n_2n_3} = \omega_1\left(n_1 + \tfrac{1}{2}\right) - \omega_2\left(n_2 + \tfrac{1}{2}\right) + \omega_3\left(n_3 + \tfrac{1}{2}\right)\,.$$

The presence of a minus sign makes the existence of level crossings plausible, and this is indeed confirmed by numerical computations. For instance, the levels $(0,0,1)$ and $(6,1,0)$ cross for a value of approximately $(|e|B/2mc) \times 2.664\,8327$ which falls inside the confinement domain.

Case B. A generalization arises if $B \neq 0,\ B_0 \neq 0$ and $V_0 = V_1 = 0$ (the charged particle evolving now in a precessing magnetic field). For this system we have performed a numerical study whose result is shown in figure 2: there are two confinement regions in the $\alpha - \alpha_0$ plane ($\alpha_0 = |e|B_0/2mc\omega$), labelled as T_1 and T_2. Note that when $B_0 \to 0$ ($\alpha_0 \to 0$), we recover the case discussed in case A.

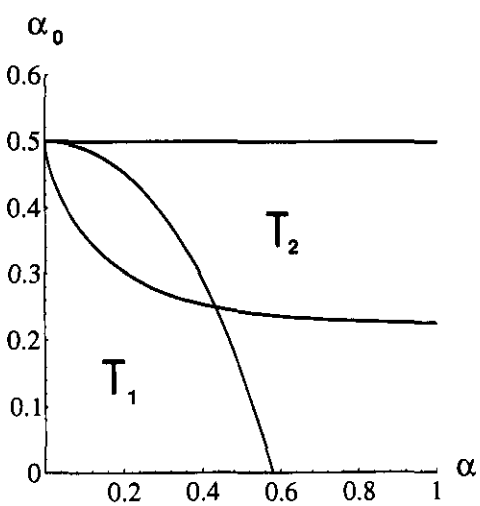

Figure 2. Division of the $\alpha - \alpha_0$ plane according to the generic motion induced by the field (2.1) on a charged particle. The regions labelled as T_1 and T_2 belong to the confinement domain, and the others are in the unconfined domain.

Case C. Another interesting physical system can be obtained making $B \neq 0$, $B_0 = 0$ and $V_0 = V_1 = m\omega_0^2/2$. With this choice of parameters, a three-dimensional charged oscillator inside a rotating magnetic field is being considered. This system has proved successful in the analysis of the resonances for a bounded system immersed in a particular kind of external radiation. Its confinement regions (marked as T_1, T_2 and T_3 in figure 3) were previously evaluated in [10], and are represented on the $\alpha - w$ plane of figure 3, $w = \omega_0/\omega = \sqrt{\gamma}_0$. Once again, in the limit $\omega_0 \to 0$ $(w \to 0)$ we recover the case A.

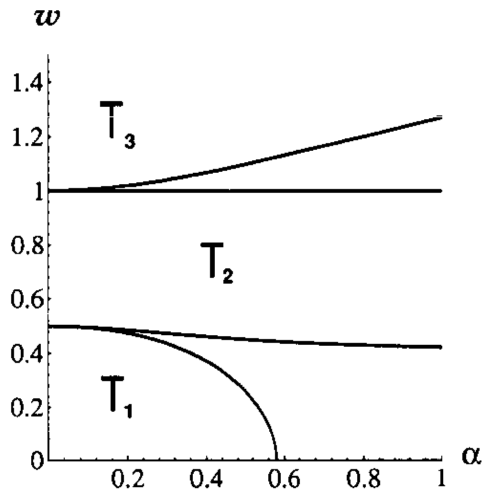

Figure 3. Plot of the domains in the $\alpha - w$ plane for the case C discussed in section 4, where a rotating magnetic field produces 'oscillating' motions in the rotating frame. Regions T_1, T_2 and T_3 form the confinement domain.

Case D. A further generalization encompassing the three previous cases is obtained if $B \neq 0$, $B_0 \neq 0$, $V_0 = V_1 = m\omega_0^2/2$ (the charged oscillator in the precessing magnetic field). The confinement regions will be volume elements which could be represented in (α, α_0, w) coordinates. In the limit when $B_0 \to 0$ $(\alpha_0 \to 0)$ these volume elements will coincide with the two-dimensional regions T_1, T_2 and T_3 of figure 3 (case C), while when $\omega_0 \to 0$ $(w \to 0)$ that domain will go into the two confinement regions T_1 and T_2 of figure 2 (case B).

In all these examples, the applicability of the techniques of sections 2 and 3 is restricted by the corresponding confinement domains: when the motion is confined, the AA phase is given simply by (3.10), (3.11). An interesting question to be explored is the behaviour of the AA phases when $\omega \to 0$. This can be performed globally for all the cases if we consider just the adiabatic limit for case D. So an explicit calculation provides the following adiabatic frequencies:

$$\begin{aligned} &\omega_1^{\mathrm{a}} = \omega_0 \qquad \omega_2^{\mathrm{a}} = \sqrt{b^2 + b_0^2 + \omega_0^2} + \sqrt{b^2 + b_0^2} \\ &\omega_3^{\mathrm{a}} = \sqrt{b^2 + b_0^2 + \omega_0^2} - \sqrt{b^2 + b_0^2} \end{aligned} \tag{4.1}$$

where the superscript a is used to indicate the adiabatic nature of the expressions (4.1). The $\partial\omega_i/\partial\omega$ are in this limit

$$\frac{\partial\omega_1^{\mathrm{a}}}{\partial\omega}=0 \qquad \frac{\partial\omega_2^{\mathrm{a}}}{\partial\omega}=\frac{-b_0}{\sqrt{b^2+b_0^2}} \qquad \frac{\partial\omega_3^{\mathrm{a}}}{\partial\omega}=\frac{b_0}{\sqrt{b^2+b_0^2}}. \tag{4.2}$$

The other cases are obtained by particularizing the values of some parameters for each example. As we can see from (3.10) and (4.1), (4.2), when $\omega_0 \to 0$ two contributions to the adiabatic phase, $\partial\omega_i^{\mathrm{a}}/\partial\omega, (i=1,3)$, have no sense because in this case the two roots ω_1^{a}, ω_3^{a} are zero and two degrees of freedom of the motion are deconfined, so the states are not cyclic. However, the third one produces a confined motion and its contribution to the adiabatic phase, proportional to $\partial\omega_2^{\mathrm{a}}/\partial\omega$, equals the solid angle in the parameter space of the magnetic field [5]. The situation is different if $\omega_0 \neq 0$. In this last case, all the contributions (4.2) to the adiabatic phase are well defined and two of them, as usual, are proportional to the solid angle in the parameter space.

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